Frobenius Theorem

If $M_1 \subset M_2$, M_1 a submanifold and if X, Y are vector fields on M_2 such that for all $p \in M_1$, $X(p) \in T_pM_1$, $Y(p) \in T_pM_1$, then $[X,Y]$ is also "tangent to M_1", namely $[X,Y]|_p \in T_pM_1$ for all $p \in M$
(For proof, compute in box-like coords. in nbhd. of p)

Definition: A C^∞ k-distribution on a manifold M_2 is a k-dimensional subspace of each tangent space of M_2 (i.e. a function from $D: M_2 \to$ set of k dim subspaces such that $D(p)$ is a k-dim. subspace of T_pM_2), C^∞ varying.
(Exercise: Decide what C^∞ means here).

Problem: Given a k-distribution, is there a (local) "integral submanifold" through a given point, i.e. a k-dim. submanifold ${}_pM_1$ with (1) given p, $p \in {}_pM_1$ and (2) for all $q \in {}_pM_1$ (q near p) $T_qM_1 = k$ distribution$|_q$.

First observation gives a condition: If X, Y are vector fields in the distribution (i.e. $\forall q \in M$, $X(q), Y(q) \in k$ distribution$|_q$), then $[X,Y]$ is in the distribution.

Lemma: If $X_1, \ldots, X_k$ generate the k distribution locally and if $[X_i, X_j]$ ~~is~~ $\forall i,j$ tangent to the k-distribution, then so is $[Y_i, Y_j]$ $\forall i,j$ if $Y_1, \ldots, Y_k$ generate the k distribution.

Proof: Write $Y_i = \sum f^i_k X_k$ and compute. $\square$

Def: A distribution is called <u>integrable</u> if it satisfies the condition of the previous lemma ~~for~~ in a neighborhood of each point

Exercise: If X, Y are tangent to an integrable distribution, then so is $[X, Y]$

Theorem: If a k-dim. distribution is integrable, then it has (local) integrable submanifolds.

Proof ~~later~~ follows:

The Proof of the Frobenius Theorem
(involutive $\Rightarrow$ Integrable)

We begin with the observation that the Theorem is true if $[X_i, X_j] = 0 \quad \forall i,j \in \{1, \dots, k\}$. {We are using our standard notation, that the k-distribution $=$ span $(X_1, \dots, X_k)$}. For this, we use the following lemma:

Lemma: If $X_1, \dots, X_k$ are linearly independent at p and if $[X_i, X_j] \equiv 0$ in a nbhd of p $(\forall i,j)$, then $\exists$ a coordinate system $(x_1, \dots, x_n)$ in a neighborhood of p such that $\forall i \in \{1, \dots, k\}$

$$X_i \equiv \frac{\partial}{\partial x_i} \quad \text{at each point of the}$$

neighborhood.

Proof: This is just a variant of our previous observation about a single vector field that has value $\neq 0$ at p being a coordinate vector field in a neighborhood of p for some coordinate system. Specifically, pick coordinates $(y_1, \dots, y_n)$ in a neighborhood of p such that $y_j(p) = 0 \ \forall j$ and $X_1(p), \dots, X_k(p), \left.\frac{\partial}{\partial y_{k+1}}\right|_p, \dots, \left.\frac{\partial}{\partial y_n}\right|_p$ are linearly independent at p. Then define a coordinate system $(x_1, \dots, x_n) \to M$ by

$(x_1, \dots, x_n) =$ the point obtained from

the point with y coordinates $(0, \cdots 0, x_{k+1}, \cdots x$
by ~~following~~ moving from that point
by x_1 along the X_1 integral curve, then by x_2 along the X_2 integral curve, .. etc.
In flow notation (φ^j = flow of X_j)

$$(x_1, \dots, x_n) = \varphi^k_{x_k}\left(\dots \varphi^2_{x_2}\left(\varphi^1_{x_1}\left(\overbrace{0, \dots, 0, x_{k+1}, \dots x_n}^{\text{point in } y \text{ coordinates}}\right)\right)\right)$$

The Inverse Function Theorem shows these are in fact coordinates for a neighborhood of p. And the fact that the flows commute easily gives $\frac{\partial}{\partial x_j} = X_j$, $\forall j \in \{1, \dots, k\}$. $\square$

Now suppose we are given an involutive k-distribution spanned say by $Y_1, \dots, Y_k$ at each point. We shall show how to transform the Y's into a new set of k generating vector fields for the distribution which have all Lie brackets $= 0$. The Lemma then gives the Theorem since under the circumstance of the Lemma the coordinate k-planes

$$(\underbrace{\cdot, \cdot, \dots, \cdot}_{\text{varying}}, \underbrace{x_{k+1}, \dots x_n}_{\text{fixed}}) \quad \text{are integral}$$

submanifolds of the k-distribution generated by $X_1, \dots X_k$ $\left(= \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right)$.

We can suppose without loss of generality that in some coordinate system $(y_1, \ldots, y_k, y_{k+1}, \ldots y_n)$ around p

$$Y_j(p) = \frac{\partial}{\partial y_j}\Big|_p \qquad j = 1, \ldots, k$$

Then for all q in a neighborhood of p

$$Y_j(q) = \sum_{l=1}^{n} f_j^l \frac{\partial}{\partial y_l} \qquad j \in \{1, \ldots, k\}$$

and the $k \times k$ matrix f_j^l $j, l \in \{1, .. k\}$ is non-singular (since it = the $k \times k$ identity at p!). Thus we can replace the Y's by linear combinations of themselves to get $Z_1, \ldots, Z_k$ with $\operatorname{span}(Z_1, \ldots, Z_k) = \operatorname{span}(Y_1, \ldots, Y_k)$ and with

$$Z_j(q) = \frac{\partial}{\partial x_j} + \sum_{l=k+1}^{n} g_j^l \frac{\partial}{\partial y_l}$$

(by multiplying by the inverse of (f_j^l) $j, l \in \{1, \ldots, k\}$)

Of course, the Z_j still have $[Z_i, Z_j] \in \operatorname{span}(Z_1, \ldots, Z_k)$, $\forall\, i, j \in \{1, \ldots k\}$.

But

$$[Z_i, Z_j] = \text{a linear combination}$$

of $\frac{\partial}{\partial y_l}$ $l \geq k+1$ since Z_i and Z_j have constant coefficients for all components of index k or lower. So for $[Z_i, Z_j]$ to be in $\operatorname{span}(Z_1, \ldots, Z_j)$, it must be that $[Z_i, Z_j] = 0$. $\square$

An example may clarify this argument.
Suppose $X_1 = \frac{\partial}{\partial x_1}$ and $X_2 = f\frac{\partial}{\partial x_1} + g\frac{\partial}{\partial x_2} + h\frac{\partial}{\partial x_3}$

on $\mathbb{R}^3$, g nowhere vanishing. ~~That~~ Then

$\frac{\partial}{\partial x_1}$ and $g\frac{\partial}{\partial x_2} + h\frac{\partial}{\partial x_3}$ span the same 2 distribution as do
$\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2} + \frac{h}{g}\frac{\partial}{\partial x_3}$.

If $\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + \frac{h}{g}\frac{\partial}{\partial x_3}\right] = \frac{\partial}{\partial x_1}\left(\frac{h}{g}\right)\frac{\partial}{\partial x_3}$

is to be in span$\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2} + \frac{h}{g}\frac{\partial}{\partial x_3}\right)$ then

it must be that $[\ ,\ ] = 0$ since otherwise there would be an $\frac{\partial}{\partial x_1}$ or a $\frac{\partial}{\partial x_2}$ component.

Thus the 2-distribution if involutive is actually generated by two vector fields with Lie bracket $= 0$.